

The Multidimensional Berry-Hannay Model

S. GUREVICH and R. HADANI

School of Mathematical Sciences

Tel-Aviv University

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Abstract

The aim of this paper is to construct the Berry-Hannay model of quantum mechanics on a $2n$ -dimensional symplectic torus. We construct a simultaneous quantization of the algebra \mathcal{A} of functions on the torus and the linear symplectic group $G = \mathrm{Sp}(2n, \mathbb{Z})$. In the construction we use the quantum torus $\mathcal{A}_{\varepsilon, \hbar}$, which is a deformation of \mathcal{A} , together with a G -action on it. We obtain the quantization via the action of G on the set of equivalence classes of irreducible representations of $\mathcal{A}_{\varepsilon, \hbar}$. For $\hbar \in \mathbb{Q}$ this action has a unique fixed point. This gives a canonical projective equivariant quantization. There exists a Hilbert space on which both G and $\mathcal{A}_{\varepsilon, \hbar}$ act in a compatible way.

0 Introduction

0.1 Motivation

In the paper “*Quantization of linear maps on the torus - Fresnel diffraction by a periodic grating*”, published in 1980 (see [BH]), the physicists M.V. Berry and J. Hannay explore a model for quantum mechanics on the 2-dimensional

torus. One of the motivations was to study the phenomenon of quantum chaos in this model (see [R] for a survey).

Berry and Hannay suggested to quantize simultaneously the functions on the torus and the linear symplectic group $\mathrm{Sp}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$.

In this paper we want to extend our construction [GH] of the 2-dimensional Berry-Hannay model to the higher dimensional tori. The central question is whether there exists a vector space on which a deformation of the algebra of functions and the linear symplectic group $\mathrm{Sp}(2n, \mathbb{Z})$, both act in a compatible way. Previously it was shown by Bouzouina and De Bievre (see [BDB]) that one can quantize simultaneously the functions on the torus and one ergodic element $A \in \mathrm{Sp}(2n, \mathbb{Z})$ in case when the Planck constant is of the form $\hbar = \frac{1}{N}$, $N \in \mathbb{N}$.

0.2 Definitions

0.2.1 Equivariant quantization of the torus

Fix a $2n$ -dimensional vector space V over \mathbb{R} and a rank $2n$ lattice $\Lambda \subset V$. Let $\mathbf{T} = V/\Lambda$ be the $2n$ -dimensional torus determined by Λ . We fix a skew-symmetric bilinear form ω on V , which we consider as a differential form on \mathbf{T} ; we assume that $\mathrm{vol}(\mathbf{T}) = 1$.

Denote by G the group $\mathrm{Sp}(\mathbf{T})$ of all the automorphisms of V which takes Λ into itself and preserve the form ω .

Let $\mathcal{A} = C^\infty(\mathbf{T})$ be the algebra of smooth complex valued functions on \mathbf{T} , and let $\Lambda^* := \{\xi \in V^* \mid \xi(\Lambda) \subset \mathbb{Z}\}$ be the dual lattice of Λ . We use the lattice Λ^* in order to have a canonical basis for \mathcal{A} . Let \langle, \rangle be the pairing between V and V^* . The map $\xi \mapsto s(\xi)$ where $s(\xi)(x) := e^{2\pi i \langle x, \xi \rangle}$, $x \in \mathbf{T}$, $\xi \in \Lambda^*$ defines a canonical isomorphism between Λ^* and the group $\mathbf{T}^* := \mathrm{Hom}(\mathbf{T}, \mathbb{C}^\times)$ of characters of \mathbf{T} . By Fourier theory the last group form a basis to \mathcal{A} .

We will construct a particular type of quantization procedure for the functions (see also [GH]). Moreover this quantization will be equivariant with respect to the action of the group of “classical symmetries” G :

Definition 0.1 *A Weyl quantization of \mathcal{A} is a family of \mathbb{C} -linear mor-*

phisms $\pi_h : \mathcal{A} \longrightarrow \text{End}(W_h)$, $\hbar \in \mathbb{R}$ s.t the following property hold:

$$\pi_h(s(\xi))\pi_h(s(\eta)) = e^{2\pi i \hbar \omega(\eta, \xi)} \pi_h(s(\eta))\pi_h(s(\xi)) \quad (1)$$

for all $\xi, \eta \in \Lambda^*$ and $\hbar \in \mathbb{R}$. Here the form ω denote the form on V^* induced by the symplectic form on V .

Definition 0.2 *An equivariant quantization of \mathbf{T} is a quantization of \mathcal{A} with additional maps $\rho_h : G \longrightarrow \text{GL}(W_h)$ s.t. the following equivariant property (called "Egorov identity") holds:*

$$\rho_h(B)^{-1} \pi_h(f) \rho_h(B) = \pi_h(f \circ B) \quad (2)$$

for all $\hbar \in \mathbb{R}$, $f \in \mathcal{A}$ and $B \in G$. If (ρ_h, W_h) is a projective representation of the group G then we call the quantization **projective**.

0.3 Results

In this paper we give an affirmative answer to the existence of the quantization procedure and give explicit formulas. We show a construction (Theorem 0.3, Corollary 0.4) of the quantization procedure for rational Planck constants. This is the first construction of such equivariant quantization for higher dimensional tori.

The idea of the construction is as follows: We use a "deformation" of the algebra \mathcal{A} of functions on \mathbf{T} . We define (see 1.1) two algebras $\mathcal{A}_{\varepsilon, \hbar}$, $\varepsilon = 0, 1$. The algebra $\mathcal{A}_{0, \hbar}$ is the usual Rieffel's quantum torus (see [Ri]) and $\mathcal{A}_{1, \hbar}$ is some twisted version of it. If $\hbar = \frac{M}{N} \in \mathbb{Q}$, then we will see that all irreducible representations of $\mathcal{A}_{\varepsilon, \hbar}$ have dimension N^n . We denote by $\text{Irr}(\mathcal{A}_{\varepsilon, \hbar})$ the set of equivalence classes of irreducible algebraic representations of the quantized algebra. We will see that $\text{Irr}(\mathcal{A}_{\varepsilon, \hbar})$ is a set "equivalent" to a torus.

The group G naturally acts on a quantized algebra $\mathcal{A}_{\varepsilon, \hbar}$ and hence on the set $\text{Irr}(\mathcal{A}_{\varepsilon, \hbar})$.

Let $\hbar = \frac{M}{N}$ with $\text{gcd}(M, N) = 1$. Set $\varepsilon = MN \pmod{2}$. Then:

Theorem 0.3 (Canonical equivariant representation) *There exist a unique (up to isomorphism) irreducible representation (π, W) of $\mathcal{A}_{\varepsilon, \hbar}$ for which its equivalence class is fixed by G (i.e $\pi \simeq^B \pi$ for all $B \in G$).*

We will give formulas for the canonical representation in section 2.

Since the canonical representation (π, W) is irreducible, by Schur's lemma we get the canonical projective representation of G compatible with π :

Corollary 0.4 (Canonical projective representation) *For every $B \in G$ there exist an operator $\rho(B)$ on W s.t:*

$$\rho(B)^{-1}\pi(f)\rho(B) = \pi(f \circ B) \quad \text{for all } f \in \mathcal{A} \quad (3)$$

Moreover the correspondence $B \mapsto \rho(B)$ gives a projective representation of G .

We will give formulas for the projective representation in section 2.

Remark 0.5 *The family $(\rho_{\hbar}, \pi_{\hbar}, W_{\hbar})$, $\hbar \in \mathbb{Q}$ presented in Corollary 0.4 gives a canonical equivariant quantization of the torus. We can endow (see 1.3) the space W_{\hbar} with a canonical unitary structure s.t π_{\hbar} is a $*$ -representation and ρ_{\hbar} unitary. This answer the question whether this quantization is also unitarizable and hence fits to the idea that quantum mechanics should be realized on a Hilbert space.*

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1 Construction

Set $\mathcal{A} = C^{\infty}(\mathbf{T})$ the algebra of smooth complex valued function on the torus. Let $\Lambda^* := \{\xi \in V^* \mid \xi(\Lambda) \subset \mathbb{Z}\}$. Let \langle, \rangle be the pairing between V and V^* .

The map $\xi \mapsto s(\xi)$ where $s(\xi)(x) := e^{2\pi i \langle x, \xi \rangle}$, $x \in \mathbf{T}$, $\xi \in \Lambda^*$ define a canonical isomorphism between Λ^* and the group $\mathbf{T}^* := \text{Hom}(\mathbf{T}, \mathbb{C}^\times)$ of characters of \mathbf{T} .

1.1 The quantum tori

Fix $\hbar \in \mathbb{R}$. Define two algebras (see also [Ri] and [GH]) $\mathcal{A}_{\varepsilon, \hbar}$, $\varepsilon = 0, 1$ as follows. The algebra $\mathcal{A}_{\varepsilon, \hbar}$ is defined over \mathbb{C} by generators $\{s(\xi), \xi \in \Lambda^*\}$, and relations:

$$s(\xi + \eta) = (-1)^{\varepsilon \omega(\xi, \eta)} e^{\pi i \hbar \omega(\xi, \eta)} s(\xi) s(\eta) \quad (4)$$

for all $\xi, \eta \in \Lambda^*$.

1.2 Weyl quantization

To get a Weyl quantization of \mathcal{A} we use a specific one-parameter family of representations (see subsection 1.4 below) of the quantum tori. This define an operator $\pi(s(\xi))$ for every $\xi \in \Lambda^*$. We extend the construction to every function $f \in \mathcal{A}$ using Fourier theory. Suppose

$$f = \sum_{\xi \in \Lambda^*} a_\xi s(\xi) \quad (5)$$

is its Fourier expansion. Then we define its *Weyl quantization* by

$$\pi(f) = \sum_{\xi \in \Lambda^*} a_\xi \pi(s(\xi)) \quad (6)$$

The convergence of the last series is due to the rapid decay of the fourier coefficients of f .

1.3 Equivariant quantization

We describe a strategy how to get an equivariant quantization of \mathbf{T} . The group $G = \text{Sp}(\mathbf{T})$ acts on Λ preserving ω . Hence G acts on $\mathcal{A}_{\varepsilon, \hbar}$. The formula of this action is ${}^B s(\xi) := s(B\xi)$. Suppose (π, W) is a representation of $\mathcal{A}_{\varepsilon, \hbar}$. For an element $B \in G$, define ${}^B \pi(s(\xi)) := \pi({}^{B^{-1}} s(\xi))$. This formula defines an action of G on the set $\text{Irr}(\mathcal{A}_{\varepsilon, \hbar})$ of equivalence classes of irreducible algebraic representations of $\mathcal{A}_{\varepsilon, \hbar}$.

Lemma 1.1 *All irreducible representations of $\mathcal{A}_{\varepsilon, \hbar}$ are N^n -dimensional.*

Suppose (π, W) is a representation for which its equivalence class is fixed by the action of G . This means that for any $B \in G$ we have $\pi \simeq {}^B\pi$, so by definition there exist an operator $\rho(B)$ on W s.t:

$$\rho(B)^{-1} \pi(s(\xi)) \rho(B) = \pi(s(B\xi)), \quad \text{for all } \xi \in \Lambda^* \quad (7)$$

This imply the Egorov identity (2) for any function. Suppose in addition that (π, W) is an irreducible representation. Then by Schur's lemma for every $B \in G$ the operator $\rho(B)$ is uniquely defined up to a scalar. This implies that (ρ, W) is a projective representation of G .

1.4 The canonical equivariant quantization

In what follows we consider only the case $\hbar \in \mathbb{Q}$. We write \hbar in the form $\hbar = \frac{M}{N}$ with $\gcd(M, N) = 1$. Set $\varepsilon = MN \pmod{2}$.

Proposition 1.2 *There exist a unique $\pi \in \text{Irr}(\mathcal{A}_{\varepsilon, \hbar})$ which is a fixed point for the action of G .*

1.5 Unitary structure

Note that $\mathcal{A}_{\varepsilon, \hbar}$ becomes a $*$ -algebra by the formula $s(\xi)^* := s(-\xi)$. Let (π, W) be the canonical representation of $\mathcal{A}_{\varepsilon, \hbar}$.

Proposition 1.3 *There exist a canonical (unique up to scalar) unitary structure on W for which π is a $*$ -representation.*

2 Formulas

We give formulas for the canonical projective equivariant quantization (π, ρ, W) . This type of realization is called the *Schrodinger model*.

Choose a polarization $\Lambda^* = L \oplus L'$ with $\omega|_L = \omega|_{L'} = 0$. The form ω defines a non-degenerate pairing between L and L' given by $\langle \xi, \xi' \rangle = \omega(\xi, \xi')$. Set $X = L/NL$ and $W = \mathcal{F}(X)$ the space of functions on X . Denote by ψ the additive character of $\mathbb{Z}/N\mathbb{Z}$ given by $\psi(t) = e^{2\pi i \hbar t}$.

2.1 Coordinates free formulas

2.1.1 Formula for π

The representation π is given by:

$$[\pi(\xi)f](x) = f(x + \xi) \quad (8)$$

$$[\pi(\xi')f](x) = \psi(< x, \xi' >)f(x) \quad (9)$$

for every $\xi \in L$, $\xi' \in L'$, $x \in X$ and $f \in \mathcal{F}(X)$.

The formula for general element is given by:

$$[\pi(\xi + \xi')f](x) = \alpha(\xi, \xi')\psi(< x, \xi' >)f(x + \xi) \quad (10)$$

where $\alpha(\xi, \xi') := (-1)^{\varepsilon\langle \xi, \xi' \rangle} e^{\pi i \hbar \langle \xi, \xi' \rangle}$.

2.1.2 Formula for ρ

The projective representation ρ is described by the following formulas:

$$\left[\rho \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} f \right](x) = f(A^{-1}x) \quad (11)$$

for every $A \in \text{GL}(L)$ - the group of invertible operators which preserve the lattice L . Here ${}^t A$ denote the operator on L' dual to A .

$$\left[\rho \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} f \right](x) = (-1)^{M\langle x, Bx \rangle} e^{\pi i \hbar \langle x, Bx \rangle} f(x) \quad (12)$$

for every $B : L \longrightarrow L'$ symmetric bilinear form.

And for every non-degenerate symmetric bilinear form $B : L \longrightarrow L'$,

$$\left[\rho \begin{pmatrix} 0 & -B^{-1} \\ B & 0 \end{pmatrix} f \right](x) = \widehat{f}(x) \quad (13)$$

where \widehat{f} denote the Fourier transform $\widehat{f}(x) := \frac{1}{\sqrt{N^n}} \sum_{y \in X} f(y) \psi(< x, By >)$.

2.2 Formulas with coordinates

Choose bases (e_1, \dots, e_n) and (e'_1, \dots, e'_n) for L and L' respectively s.t $\omega(e_i, e'_j) = \delta_{ij}$. This allows us to identify Λ^* with $\mathbb{Z}^n \oplus \mathbb{Z}^n$, the set X with $\mathbb{Z}^n / N\mathbb{Z}^n$ and the group $G = \text{Sp}(\mathbf{T})$ with $\text{Sp}(2n, \mathbb{Z})$.

2.2.1 Formula for π

The representation π is given by:

$$[\pi(\mathbf{m}, \mathbf{n})f](\mathbf{x}) = \alpha(\mathbf{m}, \mathbf{n})\psi(\langle \mathbf{x}, \mathbf{n} \rangle)f(\mathbf{x} + \mathbf{m}) \quad (14)$$

for every $\mathbf{m} \in L$, $\mathbf{n} \in L'$, $\mathbf{x} \in X$ and $f \in \mathcal{F}(X)$, where $\alpha(\mathbf{m}, \mathbf{n}) := (-1)^{\varepsilon\langle \mathbf{m}, \mathbf{n} \rangle} e^{\pi i \hbar \langle \mathbf{m}, \mathbf{n} \rangle}$.

2.2.2 formula for ρ

The projective representation ρ is described by the following formulas:

$$\left[\rho \begin{pmatrix} A & 0 \\ 0 & {}_t A^{-1} \end{pmatrix} f \right](\mathbf{x}) = f(A^{-1}\mathbf{x}) \quad (15)$$

for every $A \in \text{GL}(n, \mathbb{Z})$.

$$\left[\rho \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} f \right](\mathbf{x}) = (-1)^{M\langle \mathbf{x}, B\mathbf{x} \rangle} e^{\pi i \hbar \langle \mathbf{x}, B\mathbf{x} \rangle} f(\mathbf{x}) \quad (16)$$

for every symmetric matrix $B \in \text{Mat}(n, \mathbb{Z})$.

And for every invertible symmetric matrix $B \in \text{Mat}(n, \mathbb{Z})$:

$$\left[\rho \begin{pmatrix} 0 & -B^{-1} \\ B & 0 \end{pmatrix} f \right](\mathbf{x}) = \widehat{f}(\mathbf{x}) \quad (17)$$

where \widehat{f} denote the Fourier transform as defined in 2.1.2.

3 Proofs

3.1 Proof of Proposition 1.1

Suppose (π, W) is an irreducible representation of $\mathcal{A}_{\varepsilon, \hbar}$.

Step 1. First we show that W is finite dimensional. The algebra $\mathcal{A}_{\varepsilon, \hbar}$ is a finite module over $Z(\mathcal{A}_{\varepsilon, \hbar}) = \{s(N\xi), \xi \in \Lambda^*\}$ which is contained in the center of $\mathcal{A}_{\varepsilon, \hbar}$. Because W has at most countable dimension (as a quotient space of $\mathcal{A}_{\varepsilon, \hbar}$) and \mathbb{C} is uncountable then by Kaplansky's trick (See [MR]) $Z(\mathcal{A}_{\varepsilon, \hbar})$ acts on W by scalars. Hence $\dim W < \infty$.

Step 2. We show that W is N^n -dimensional.

Choose a basis $(e_1, \dots, e_n, e'_1, \dots, e'_n)$ of Λ^* s.t $\omega(e_i, e_j) = \omega(e'_i, e'_j) = 0$ and $\omega(e_i, e'_j) = \delta_{ij}$ the Kronecker's delta. Denote by E the commutative subalgebra of $\mathcal{A}_{\varepsilon, \hbar}$ generated by $\{s(e_i)\}_1^n$. Suppose $\lambda \in E^*$ is an eigencharacter of E and denote by $W_\lambda = W_{(\lambda_1, \dots, \lambda_n)}$ the corresponding eigenspace, $\lambda_i := \lambda(e_i)$. We have the following commutation relation $\pi(e_i)\pi(e'_j) = \gamma^{\delta_{ij}}\pi(e'_j)\pi(e_i)$ where $\gamma := e^{-2\pi i \frac{M}{N}}$. Hence $\pi(e'_j) : W_{(\lambda_1, \dots, \gamma^k \lambda_j, \dots, \lambda_n)} \longrightarrow W_{(\lambda_1, \dots, \gamma^{k+1} \lambda_j, \dots, \lambda_n)}$. Since $\gcd(M, N) = 1$ the eigencharacters $(\gamma^{k_1} \lambda_1, \dots, \gamma^{k_n} \lambda_n)$, $0 \leq k_j \leq N-1$ are all different. Let $0 \neq w \in W_\lambda$. Recall that $\pi(e'_j)^N = \pi(Ne'_j)$ is a scalar operator. The space $\text{span}\{\pi(e'_j)^{k_j} w\}$ is N^n -dimensional $\mathcal{A}_{\varepsilon, \hbar}$ -invariant subspace hence equal W . \square

3.2 Proof of Proposition 1.2

Suppose (π, W) is an irreducible representation of $\mathcal{A}_{\varepsilon, \hbar}$. By Schur's lemma for every $\xi \in \Lambda^*$, $\pi(N\xi)$ is a scalar operator $\pi(N\xi) = \chi_\pi(\xi) \cdot \text{I}$. We have $\pi(0) = \text{I}$, hence $\chi_\pi(\xi) \neq 0$ for all $\xi \in \Lambda^*$. Thus to any irreducible representation we have attached a scalar function $\chi_\pi : \Lambda^* \longrightarrow \mathbb{C}^\times$. It is easy to see that $\chi_\pi(\xi + \eta) = \chi_\pi(\xi)\chi_\pi(\eta)$. Let $\mathbb{T}(\mathbb{C}) := \text{Hom}(\Lambda^*, \mathbb{C}^\times)$ be the group of complex characters of Λ^* . We have defined a map $\text{Irr}(\mathcal{A}_{\varepsilon, \hbar}) \longrightarrow \mathbb{T}(\mathbb{C})$ given by $\pi \mapsto \chi_\pi$. This map is obviously compatible with the action of G , where the group G acts on characters by ${}^B\chi(\xi) := \chi(B^{-1}\xi)$.

Lemma 3.1 *The map $\pi \mapsto \chi_\pi$ gives a G -equivariant bijection:*

$$\text{Irr}(\mathcal{A}_{\varepsilon, \hbar}) \longrightarrow \mathbb{T}(\mathbb{C})$$

From Lemma 3.1 we easily deduce that there exists a unique equivalence class $\pi \in \text{Irr}(\mathcal{A}_{\varepsilon, \hbar})$ which is fixed by the action of G . This is the one that corresponds to the trivial character $\mathbf{1} \in \mathbb{T}(\mathbb{C})$ which is the unique fixed point for the action of G on $\mathbb{T}(\mathbb{C})$.

Proof of Lemma 3.1. *Step 1.* The map $\pi \mapsto \chi_\pi$ is onto. We define an action of $\mathbb{T}(\mathbb{C})$ on $\text{Irr}(\mathcal{A}_{\varepsilon, \hbar})$ and on itself by $\pi \mapsto \chi\pi$ and $\psi \mapsto \chi^N\psi$, where

$\chi \in \mathbb{T}(\mathbb{C})$, $\pi \in \text{Irr}(\mathcal{A}_{\varepsilon, \hbar})$ and $\psi \in \mathbb{T}(\mathbb{C})$. The map $\pi \mapsto \chi_\pi$ is clearly a $\mathbb{T}(\mathbb{C})$ -equivariant map with respect to these actions. The claim follows since the above action of $\mathbb{T}(\mathbb{C})$ on itself is transitive.

Step 2. The map $\pi \mapsto \chi_\pi$ is one to one. Suppose (π, W) is an irreducible representation of $\mathcal{A}_{\varepsilon, \hbar}$. It is easy to deduce from the proof of Lemma 1.1 that for $\xi \notin N\Lambda^*$, $\text{tr}(\pi(\xi)) = 0$. But we know from character theory that an isomorphism class of a finite dimensional irreducible representation of an algebra is recovered from its character. \square

3.3 Proof of Proposition 1.3

Existence. We use the realization of the representation π given in section 2. It is easy to see that the representation $(\pi, \mathcal{F}(X))$ is a $*$ -representation with respect to the standard scalar product on $\mathcal{F}(X)$ defined by:

$$\langle f, g \rangle = \frac{1}{N^n} \sum_{x \in X} f(x) \overline{g(x)} \quad (18)$$

Uniqueness. From Schur's lemma follows that the Hilbert structure endowed on the vector space of the representation is unique up to a scalar. \square

Remark 3.2 *It can be shown that all the irreducible $*$ -representations of $\mathcal{A}_{\varepsilon, \hbar}$ on Hilbert spaces are the ones for which χ_π takes values in the circle S^1 .*

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Gurevich Shamgar
School of Mathematical Sciences
Tel Aviv University
Tel Aviv 69978, Israel
e-mail: gorevitc@post.tau.ac.il

Hadani Ronny
School of Mathematical Sciences
Tel Aviv University
Tel Aviv 69978, Israel
e-mail: hadani@post.tau.ac.il